(Non)compositionality in categorical systems theory

Jules Hedges[1]

[1]Max Planck Institute for Mathematics in the Sciences, Leipzig

ACT 2020, 6th July 2020, via quarantine

Categorical systems theory

Consider a category where:

- Morphisms are open systems
- Objects are boundaries

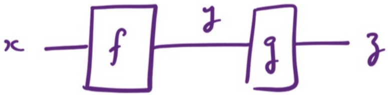
left boundary $x \xrightarrow{f} y$ right boundary

N.b. Why a category?

- i.e. why a strict separation into left and right boundaries?

We don't need a category! Other operad algebras work too!
Categories are just convenient!

Complex systems



The composition *fg* is coupling along a common boundary, and yields a complex system, i.e. a system that is a complex of smaller parts [1]

Often $\mathcal{C}$ also admits a monoidal structure for disjoint (non-coupling) composition

[1] Not to be confused with a complicated system

Systems vs. processes

Side remark:
As well as systems theories we also have process theories

Most of this talk still applies, replace “left boundary” and “right boundary” with input and output

Behaviour

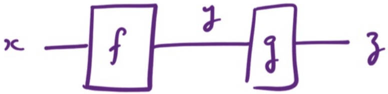
To each boundary x we associate a set $B(x)$ of possible behaviours that can be observed on that boundary

To each open system $f : x \to y$ we associate a set $B(f) \subseteq B(x) \times B(y)$

- $(a, b) \in B(f)$ means "it is possible to simultaneously observe a on the left boundary and b on the right boundary of f"

No observations can be made except on the boundary

Behaviours compose



Suppose f and g share a common behaviour on their common boundary:

$$(a, b) \in B(f) \text{ and } (b, c) \in B(g) \text{ for some } b \in B(y)$$

In most situations, this implies that $(a, c) \in B(fg)$ [2]
So:

$$B(f)B(g) \subseteq B(fg)$$

(LHS composition in **Rel**)

[2] If your behaviour doesn't satisfy this, you should probably try something else

In fancy terminology:

B is a lax functor[3] $\mathcal{C} \to \mathbf{Rel}$

- $\mathcal{C}$ is locally discrete 2-category (exactly one 2-cell)
- **Rel** is a locally thin 2-category (at most one 2-cell)

[3] Or lax pseudofunctor if being pedantic

Emergent behaviours

In many practical situations, the converse fails:

fg can exhibit "emergent" behaviours that do not arise from individual behaviours of f and g

Definition. An emergent behaviour of fg (w.r.t. the decomposition (f, g)) is an element of $B(fg) \setminus B(f)B(g)$

So we do not have a functor $B : \mathcal{C} \to \mathbf{Rel}$

In practice: This is much less interesting
Functoriality sometimes fails very badly in real examples

Grand challenge for ACT

Given a lax functor to **Rel** [4] associate some mathematical object (cohomology?) that 'describes' how it fails to be a functor (i.e. how the laxator fails to be an iso), in a useful way

"Useful" = encodes something worth knowing about emergent behaviour

[4] Or linear/additive relations etc. as convenient

Example: Open graph reachability[5]

OGph = structured cospan category of open graphs

- Objects: finite sets
- Morphisms: cospans of graph homomorphisms

$$L(x) \xrightarrow{\iota_1} g \xleftarrow{\iota_2} L(y)$$

$L(-)$ = discrete graph on a set

[5] Probably not the best example, but the easiest to draw pictures of

Reachability

Define $B : \mathbf{OGph} \to \mathbf{Rel}$ by:

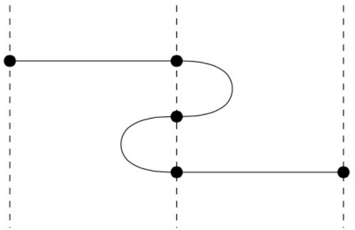
- On objects: $B(x) = x$
- On morphisms: $B(x \xrightarrow{\iota_1} g \xleftarrow{\iota_2} y) =$

$$\{(a,b) \in (x,y) \mid \iota_1(a) \text{ and } \iota_2(b) \text{ are connected in } g\}$$

Proposition. B is a lax functor

Reachability is not a functor

Minimal counterexample: the zig-zag



$$L(1) \longrightarrow f \longleftarrow L(3) \longrightarrow g \longleftarrow L(1)$$

$$B(f)B(g) = \varnothing \subsetneq \{(*, *)\} = B(fg)$$

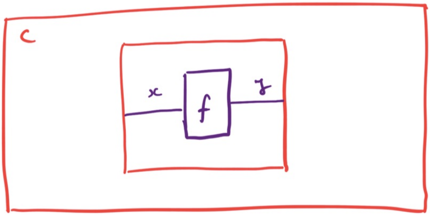
Comorphisms

Idea: It doesn't make sense to ask what is the behaviour of an open system in isolation - only in context

Call a possible context for a morphism $x \to y$ a comorphism

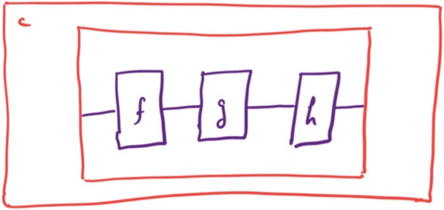
Write $\overline{\mathcal{C}}(x, y)$ for the set of comorphisms $x \to y$

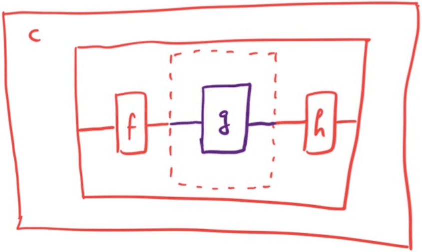
Draw them as inside-out diagram elements[6]



[6] Or alternatively as combs

$\overline{\mathcal{C}}$ is a functor $\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to$ **Set**



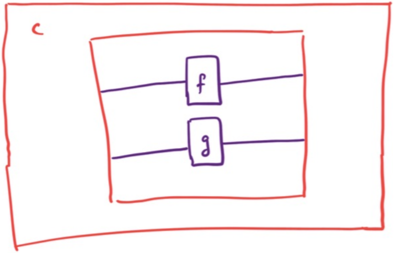


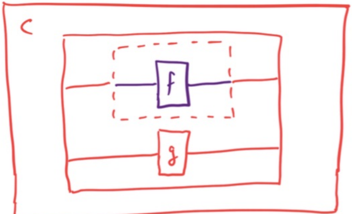
Notation: Write $\overline{\mathcal{C}}(f, \mathrm{id})(c) = f_* c$ and $\overline{\mathcal{C}}(\mathrm{id}, f)(c) = f^* c$ (so $\overline{\mathcal{C}}(f, g)(c) = f_* g^* c = g^* f_* c$)

Contexts in a monoidal category

$$/_{x_1,x_2,y_1,y_2} : \overline{\mathcal{C}}(x_1 \otimes x_2, y_1 \otimes y_2) \times \mathcal{C}(x_1, y_1) \to \overline{\mathcal{C}}(x_2, y_2)$$

$$\backslash_{x_1,x_2,y_1,y_2} : \overline{\mathcal{C}}(x_1 \otimes x_2, y_1 \otimes y_2) \times \mathcal{C}(x_2, y_2) \to \overline{\mathcal{C}}(x_1, y_1)$$



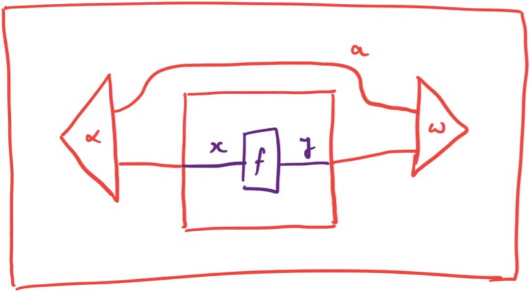


General purpose contexts 1

For $\mathcal{C}$ a monoidal category,

$$\overline{\mathcal{C}}(x, y) = \int^{a \in \mathcal{C}} \mathcal{C}(I, a \otimes x) \times \mathcal{C}(a \otimes y, I)$$

aka. a state in the category of optics, $\overline{\mathcal{C}}(x, y) \cong \mathbf{Opt}_{\mathcal{C}} \left(I, \binom{x}{y} \right)$



General purpose contexts 2

For $\mathcal{C}$ a traced monoidal category, $\overline{\mathcal{C}}(x, y) = \mathcal{C}(y, x)$

For $\mathcal{C}$ a compact closed category, $\overline{\mathcal{C}}(x, y) = \mathcal{C}(I, x \otimes y)$

Augmented semantic category

Fix a semantic category $\mathcal{D}$ (e.g. $\mathcal{D} = \mathbf{Rel}$)
and a mapping on objects $B : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$

Definition. Define a category $\widehat{\mathcal{C}}$ by:

- $\mathrm{Ob}(\widehat{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$
- $\widehat{\mathcal{C}}(x, y) = \mathcal{C}(x, y) \times (\overline{\mathcal{C}}(x, y) \to \mathcal{D}(B(x), B(y)))$

i.e. morphisms are pairs of

1. A system
2. Its behaviour in every context

- Identity: $(\mathrm{id}_x, \widehat{\mathrm{id}_x})$ where $\widehat{\mathrm{id}_x}(c) = \mathrm{id}_{B(x)}$ for all $c \in \overline{\mathcal{C}}(x, x)$
- Composition: $(f, \widehat{f})(g, \widehat{g}) = (fg, \widehat{fg})$ where $\widehat{fg}(c) = \widehat{f}(g^* c)\widehat{g}(f_* c)$

$\widehat{\mathcal{C}}$ as a semantic category

There is forgetful functor $\widehat{\mathcal{C}} \to \mathcal{C}$ [7]

We would like to find sections of it
and view $\widehat{\mathcal{C}}$ as our semantic category instead of $\mathcal{D}$

Highly dubious central claim: This is workable framework for functorial semantics, in settings with emergent effects

N.b. This is the essence of how open games work

[7] Unproven guess: Some reasonable extra hypotheses on $\overline{\mathcal{C}}$ make it a bifibration

Augmented semantic functors

Proposition. Let $B_{x,y} : \mathcal{C}(x,y) \times \overline{\mathcal{C}}(x,y) \to \mathcal{D}(B(x), B(y))$ be a family of functions satisfying

- $B_{x,x}(\mathrm{id}_x, c) = \mathrm{id}_{B(x)}$ for all $c \in \overline{\mathcal{C}}(x,x)$
- $B_{x,z}(fg, c) = B_{x,y}(f, g^* c) B_{y,z}(g, f_* c)$

This induces a section $B : \mathcal{C} \to \widehat{\mathcal{C}}$ by $f \mapsto (f, B(f, -))$.

Example: Context for reachability

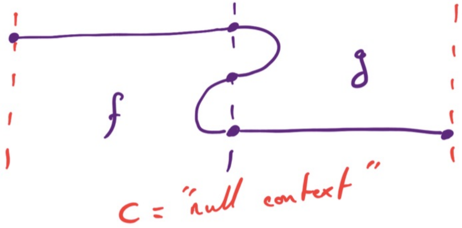
We could take $\overline{\mathbf{OGph}}$ to be e.g. the general purpose context for compact closed categories

Instead let's try something ad-hoc tailored to reachability

Let $\overline{\mathbf{OGph}}(x, y)$ to be the set of open graphs $x \to y$ whose set of nodes is exactly $x + y$

Idea: Edges in $c \in \overline{\mathbf{OGph}}(x, y)$ represent reachability in the "real" context

Reachability in context



Reachability in context