Non-compositionality in categorical systems theory

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1 Introduction

A common situation in categorical systems theory [Fon16] is that we have a category $\mathcal{C}$ whose morphisms are open systems and whose morphisms $x \to y$ are open systems with left boundary x and right boundary y. The identity morphism on an object represents a trivial system that simply identifies its boundaries with each other, and composition represents coupling (or *composing*) a pair of open systems along their common boundary to form a *complex system* (ie. a complex of systems).

To each boundary x we associate a set $F(x)$ of behaviours that could be observed on that boundary, and to each system $f : x \to y$ we associate a relation $F(f) \subseteq F(x) \times F(y)$, where $(p, q) \in F(f)$ means that it is possible to simultaneously observe the behaviours p and q on the left and right boundary of f. Under reasonable conditions it is the case that if a pair of composable systems admit some behaviours that agree on the common boundary then the composite system will also admit that behaviour: if $(p, q) \in F(f)$ and $(q, r) \in F(g)$ for some behaviour $q \in F(y)$, then $(p, r) \in F(fg)$. However, the converse is usually not the case in nontrivial settings: a composite system admits behaviours that do not arise from possible behaviours of the individual parts, usually called *emergent effects*, or sometimes 'generative effects' [Ada17, FS19].

In categorical terminology, a category of open systems usually admits a *lax* functor to **Rel**, where $\mathcal{C}$ is viewed as a 2-category with only identity 2-cells and **Rel** is viewed as a 2-category with inclusions as 2-cells. The author has proposed a 'grand challenge' for applied category theory to give a finer-grained description of lax functors to categories such as **LinRel**, for example using cohomology.

This note describes an alternative, more pragmatic approach, based on the idea that it is a mistake is to try to understand the behaviour of a system in isolation. An open system lives inside a *context*, which can influence its behaviour. But it is too weak to consider the possible behaviours in *any* context: for example, if we are allowed to consider an open graph as sitting inside any graph with a 1-hole context then every boundary vertex could be reachable from any other.

2 Contexts

Let $\mathcal{C}$ be a category, and let cohom : $\mathcal{C} \times \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a functor. We call an element $c \in \text{cohom}_{\mathcal{C}}(x, y)$ a *comorphism*, or a *context* for morphisms $x \to y$. Given $f : x \to x'$ and $g : y' \to y$, we use the shorthand notation $f_* c g^* = \text{cohom}_{\mathcal{C}}(f, g)(c) \in \text{cohom}_{\mathcal{C}}(x', y')$, as well as $f_* c = \text{cohom}_{\mathcal{C}}(f, y)(c)$ and $c g^* = \text{cohom}_{\mathcal{C}}(x, g)(c)$.

If $\mathcal{C}$ is a symmetric monoidal category then $\text{cohom}_{\mathcal{C}}$ must be a monoidal functor, and we additionally require a natural family of functions

$$/ : \text{hom}_{\mathcal{C}}(x_2, y_2) \times \text{cohom}_{\mathcal{C}}(x_1 \otimes x_2, y_1 \otimes y_2) \to \text{cohom}_{\mathcal{C}}(x_1 \otimes y_1)$$

This definition can be found in full in [Hed19].

There are some examples of generally-useful context functors, depending on the structure of $\mathcal{C}$. If $\mathcal{C}$ is (only) a monoidal category then we can take

$$\text{cohom}_{\mathcal{C}}(x, y) = \int^{a \in \mathcal{C}} \text{hom}_{\mathcal{C}}(I, x \otimes a) \times \text{hom}_{\mathcal{C}}(y \otimes a, I) = \text{hom}_{\mathbf{Opt}(\mathcal{C})}(I, (x, y))$$

where $\mathbf{Opt}(\mathcal{C})$ is the category of optics over $\mathcal{C}$ [Ril18]. This subsumes the contexts used in compositional game theory [BHZ19]. If $\mathcal{C}$ is additionally traced symmetric monoidal (for example compact closed) then we can instead take $\text{cohom}_{\mathcal{C}}(x, y) = \text{hom}_{\mathcal{C}}(y, x)$ [Hed19]. However, we can also design more specific 'custom' contexts for different situations. An example will be given in the next section.

In the following, the symbol $\dashv\!\!|$ can be vocalised as "in context".

Definition 1. *Let $\mathcal{C}$ be a category, $\text{cohom}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C}^{\text{op}} \to \mathbf{Set}$ a functor, and $\mathcal{D}$ another category with a mapping on objects $F : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$. We define a category $\mathcal{C}^{\dashv\!|}$ where:*

- *Objects of $\mathcal{C}^{\dashv\!|}$ are objects of $\mathcal{C}$*
- *Morphisms $x \to y$ in $\mathcal{C}^{\dashv\!|}$ are pairs $(f, f^{\dashv\!|})$ where $f \in \text{hom}_{\mathcal{C}}(x, y)$ and $f^{\dashv\!|} : \text{cohom}_{\mathcal{C}}(x, y) \to \text{hom}_{\mathcal{D}}(F(x), F(y))$*
- *The identity morphism on x is $(\text{id}_x, \text{id}_x^{\dashv\!|})$, where $\text{id}_x^{\dashv\!|} c = \text{id}_{F(x)}$ for all $c \in \text{cohom}_{\mathcal{C}}(x, y)$*
- *Composition is given by $(f, f^{\dashv\!|})(g, g^{\dashv\!|}) = (fg, (fg)^{\dashv\!|})$, where $(fg)^{\dashv\!|} c = (f^{\dashv\!|} c g^*)(g^{\dashv\!|} f_* c)$*

We usually imagine $\mathcal{D} = \mathbf{Rel}$ in the previous definition.

Proposition 1. *$\mathcal{C}^{\dashv\!|}$ is indeed a category.*

Trivially there is an identity-on-objects functor $U : \mathcal{C}^{\dashv\!|} \to \mathcal{C}$ given by $U(f, f^{\dashv\!|}) = f$.

Proposition 2. *Let $\dashv\!|_{x,y} : \text{hom}_{\mathcal{C}}(x, y) \times \text{cohom}_{\mathcal{C}}(x, y) \to \text{hom}_{\mathcal{D}}(F(x), F(y))$ be a family of functions satisfying*

- *$\text{id}_x \dashv\!|_{x,x} c = \text{id}_{F(x)}$ for all $c \in \text{cohom}_{\mathcal{C}}(x, x)$*
- *$fg \dashv\!|_{x,z} c = (f \dashv\!|_{x,y} c g^*)(g \dashv\!|_{y,z} f_* c)$ for all $x \xrightarrow{f} y \xrightarrow{g} z$ and $c \in \text{cohom}_{\mathcal{C}}(x, z)$*

Then this data induces a functor $\dashv\!| : \mathcal{C} \to \mathcal{C}^{\dashv\!|}$, by $f \mapsto (f, f \dashv\!| -)$.

3 Example: reachability in graphs

Let **OGph** be structured cospan category [BC19] of open (finite simple) graphs: objects are finite sets, morphisms $x \to y$ are cospans $L(x) \to g \leftarrow L(y)$ of graphs modulo cospan equivalence of graph homomorphisms, where L gives the discrete graph on a set. Reachability in open graphs yields a lax functor $F : \mathbf{OGph} \to \mathbf{Rel}$. Laxness happens due to the possibility of a path crossing a boundary multiple times; a minimal example is given in figure 1. We consider this the simplest representative of a class of similar problems, which includes reachability in Petri nets [BM20].

On objects, we take the context functor $\mathrm{cohom}_{\mathbf{OGph}} : \mathbf{OGph} \times \mathbf{OGph}^{\mathrm{op}} \to \mathbf{Set}$ to be given on objects by $\mathrm{cohom}_{\mathbf{OGph}}(x, y)$ being the set of pairs (α, ω) where $\alpha : 0 \to x$ and $\omega : y \to 0$ are open graphs whose set of vertices are exactly x and y respectively. On morphisms $f : x \to x'$ and $g : y' \to y$, we take $\mathrm{cohom}_{\mathbf{OGph}}(f, g)(\alpha, \omega) = (f_* \alpha, g^* \omega)$ where:

- $f_* \alpha : 0 \to x'$ is the open graph with set of nodes x' in which we add an edge between a pair of nodes precisely when there is a path between them in αf
- $g^* \omega : y' \to 0$ is the open graph similarly produced from $g\omega$

Proposition 3. $\mathrm{cohom}_{\mathbf{OGph}}$ *is indeed a functor* $\mathbf{OGph} \times \mathbf{OGph}^{\mathrm{op}} \to \mathbf{Set}$.

We take the mapping on objects $F : \mathrm{Ob}(\mathbf{OGph}) \to \mathrm{Ob}(\mathbf{Rel})$ to be $F(x) = x$. Define $\dashv\!\Vdash_{x,y}: \mathrm{hom}_{\mathbf{OGph}}(x, y) \times \mathrm{cohom}_{\mathbf{OGph}}(x, y) \to \mathrm{hom}_{\mathbf{Rel}}(x, y)$ by taking $f \dashv\!\Vdash_{x,y} (\alpha, \omega)$ to be the set of pairs $(p, q) \in x \times y$ such that one of the following two conditions holds:

1. $\iota_L(p) = \iota_R(q)$, where ι_L and ι_R are the legs of the cospan $f : x \to y$
2. There is a path $\iota_L(p) \rightsquigarrow \iota_R(q)$ in the (closed) graph $\alpha f \omega$ that includes at least one edge in f

Proposition 4. *This satisfies the conditions of proposition 2, and thus induces a functor* $\dashv\!\Vdash : \mathbf{OGph} \to \mathbf{OGph}^{\dashv\Vdash}$.

We illustrate this with the open graph $1 \xrightarrow{f} 3 \xrightarrow{g} 1$ depicted in figure 1. Let $(\alpha, \omega) \in \mathrm{cohom}_{\mathbf{OGph}}(1, 1)$ be the 'null' context where $\alpha : 0 \to 1$ and $\omega : 1 \to 0$ both contain no edges. (Every cohom-set contains such a distinguished 'null' context, which is additional structure present in this particular example.) The resulting context of f is shown in figure 2, and g is similar. The resulting relations $f \dashv\!\Vdash_{1,3} (\alpha, g^* \omega) \subseteq 1 \times 3$ and $g \dashv\!\Vdash_{3,1} (f_* \alpha, \omega) \subseteq 3 \times 1$ are both full relations. This is expected, because the open graph in figure 1 is connected and thus every node is reachable from every other.

Although the target of the functor is not **Rel**, there is a sense in which this gives a compositional description of the reachability relation.

References

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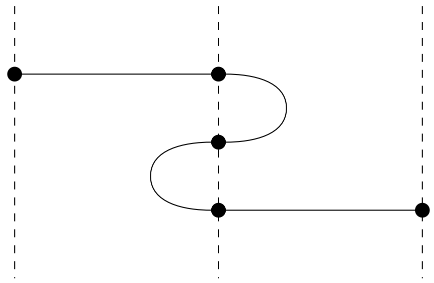
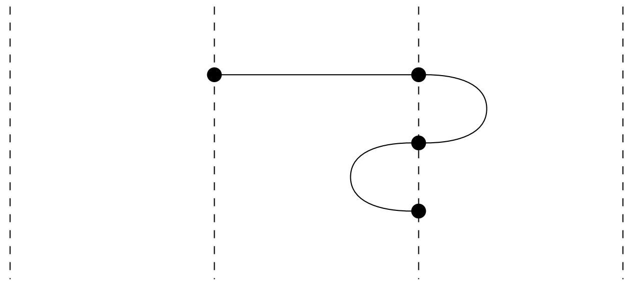


$$L(1) \longrightarrow f \longleftarrow L(3) \longrightarrow g \longleftarrow L(1)$$

Figure 1: Laxness of reachability



$$L(0) \longrightarrow \alpha \longleftarrow L(1) \longrightarrow f \longleftarrow L(3) \rightarrow g^*\omega \leftarrow L(0)$$

Figure 2: f in context

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