5. Backward induction and imperfect information

This lecture: Continuing the previous one on extensive form.

0. Recap

Extensive form, strategies, normal/strategic form, subgames, subgame perfect equilibria.

1. Backward induction

Backward induction is the method used to prove:

Theorem (Zermelo) Every finite extensive form game has a subgame perfect equilibrium.
The backward induction algorithm

Given a game tree:

1. Pick an internal node whose children are all leaf nodes.
2. Pick an available action a that maximizes the payoff for the aiming player among the descendant leaves.
3. Delete the chosen node and replace it by the leaf at the chosen action.
4. Repeat from 1 until only a single leaf remains.
Notes: We rely on finiteness twice:
(1) Every node has a maximising action because it is finitely branching.
(2) The algorithm terminates because there are finitely many nodes.

During a run of the algorithm, we choose an action at every node — i.e. a strategy profile.

Lemma (1) The strategy profile constructed by BI is a subgame perfect equilibrium.
(2) Every SPE can be constructed by BI given some choices.
Ex. The Ultimatum game again.

\begin{align*}
\text{F} & \quad \text{U} \\
\text{A} & \quad \text{R} \\
(5,5) & \quad (0,0) \\
(8,2) & \quad (0,0)
\end{align*}

We previously saw that there are 3 Nash eq. in pure strategies: (F, AR), (U, AA), (U, RA) and (U, AA) is the only SPE.

Apply BI:

\begin{align*}
\text{F} & \quad \text{U} \\
\text{A} & \quad \text{R} \\
(5,5) & \quad (0,0) \\
(8,2) & \quad (0,0)
\end{align*} 

\Rightarrow \quad \begin{align*}
\text{F} & \quad \text{U} \\
\text{A} & \quad \text{R} \\
(5,5) & \quad (8,2)
\end{align*} 

A only NE

\Rightarrow \quad \begin{align*}
\text{F} & \quad \text{U} \\
\text{A} & \quad \text{R} \\
(5,5) & \quad (8,2)
\end{align*} 

\Rightarrow \quad \begin{align*}
\text{F} & \quad \text{U} \\
\text{A} & \quad \text{R} \\
(5,5) & \quad (8,2)
\end{align*} 

\Rightarrow \quad (8,2)

U only NE
Ex. The centipede game
(because its EF tree looks like a centipede)

[ = Tausendfüßer; in English we distinguish
100 legs from 1000 legs ].

\[
\begin{array}{c|ccc|ccc}
Q & C & C & C & Q & C & C \\
\hline
Q & (3,5) & Q & Q & Q & Q & Q \\
(1,0) & (0,2) & (3,1) & (2,4) & (4,3) & & \\
\end{array}
\]

The pattern can be extended to any finite length:

\[
\begin{array}{c|c|c|c}
1 & 2 & 1 & 2 \\
\hline
(1,0) & (0,2) & (3,1) & (10,98) \\
(99,100) & (99,100) & & \\
\end{array}
\]
5. Backward induction and imperfect information

Apply backward induction:

\[
\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (3,5) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & (3,1) & (2,4) & (4,3) \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (4,3) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & (3,1) & (2,4) & \text{Q only NE} \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (2,4) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & (3,1) & \text{Q only NE} \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (3,1) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & \text{Q only NE} \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (2,4) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & \text{Q only NE} \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (1,1) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & \text{Q only NE} \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (0,2) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & \text{Q only NE} \\
\end{array}
\Rightarrow

\begin{array}{cccc|c}
\text{C} & \text{C} & \text{C} & \text{C} & (0,2) \\
\text{Q} & \text{Q} & \text{Q} & \text{Q} & \text{Q} \\
(1,0) & (0,2) & \text{Q only NE} \\
\end{array}
\Rightarrow

(1,0)
The centipede game of any length has only 1 SPE: both players quit at every node.

The catch: Humans don't play like that at all!

This is a different problem to not having "accurate" payoffs: humans just don't reason by backward induction beyond a few levels.
Ex. (chosen artificially to illustrate something that I think is weird about SPE).

This has 2 SPE: (L,L) and (R,R).

Player 1 acts as though player 2 has already committed.

In reality: Player 2 has qualitative uncertainty about what player 2 would do in the subgame.
2. Imperfect information

All of our extensive form games so far have been games of perfect information — players always know the current state of the game (i.e. node of the tree) exactly. Technically, this means that strategies are given by a choice of action for any node.

Principle (my principle, not a standard one)

— information available to players is encoded by letting strategies be functions

\[ \sigma : O \rightarrow A \]

observations
A canonical example of an imperfect information game is the market entry game.

A market has an incumbent (established firm) I and an entrant (new firm) E.

In the first round, E can either quit (Q) and the game terminates, or continue (C). If E chooses C then I and E play a trimactric game where they can fight (F) or accommodate (A), (for example by setting aggressively low prices to fight).
In formally, this is the situation:

\[ \begin{array}{c|cc}
   & F & A \\
  \hline
  E & (-3, -1) & (1, -2) \\
  A & (-2, -1) & (3, 1) \\
\end{array} \]

Let's work out the pure strategy Nash equilibria of the subgame:

\[ (A, A) \text{ only} \]

NE in pure strategies.
So by backward induction we expect to reduce the tree to

\[
\begin{array}{c}
\text{E} \\
\uparrow \\
\text{C} \\
(0,2) \\
(3,1)
\end{array}
\]

leading to the unique SPE \((C,A,A)\).

Formally, extensive + normal form representations aren’t usually combined like this (sadly, I think it’s a better representation). The “official” representation uses information sets.

**Def.** An extensive form game of imperfect information is an extensive form, together with an equivalence relation \(\sim\) on the internal nodes, such that

1. if \(x \sim y\) then \(x, y\) are owned by the same player.
2. if \(x \sim y\) then the set of actions available is the same, \(A(x) = A(y)\).
Equivalence classes of $n$ are called information sets.

Idea: When the game is in state (node) $x$, the owning player knows that they are somewhere in $[x]$, but don't know where. This means that their strategy must be constant on every information set they own.

Def. A pure strategy for a player in an imperfect information game is a choice of action at every information set they own.

Notice that the 2 conditions we imposed are a mole his well-defined.
We draw the market entry game with information sets like this:

The dotted line denotes equivalence.

So here are 3 information sets:

$\{a, b, 3\}, \{c, d\}$.

We could equivalently draw the same game:

NB: The question of when 2 games are "the same" is very difficult and subtle!
In this way we can equivalently depict any normal form game in extensive form, e.g.

```
   p1
   /\    (Matching pennies)
  H /  T
 /   /    \
H | T | H | T
```

\[(1,0) \quad (0,1) \quad (0,1) \quad (1,0)\]

This example shows that Zermelo's theorem (existence of SPE \( \rightarrow \) existence of pure NE) doesn't hold for imperfect information.
We can still define normal forms as before (strategies become choices), so we get:

**Corollary of Nash's Theorem**: Every finite extensive form game of imperfect information has a Nash equilibrium in mixed strategies.

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Literally: a probabilistic mixture of pure strategies. (All probability is up front, not during play)

Combining mixed Nash with SPE is subtle — what should we do in a subgame that was assigned probability 0?

We'll come back to this in a few weeks!
Def. A subgame of an extensive form game is a subtree such that if \( x \) is in the subgame and \( xny \) then \( y \) is in the subgame. (\( x \)) with inherited structure, including information sets.

Now there are non-trivial games with no proper subgames, e.g.

\[ \begin{array}{c}
\text{Theorem} \\
\text{In a finite extensive form game of imperfect information, every SPE (there may be none) can be found by generalised backward induction: recursively replace every "minimal subgame" by the leaf resulting from some pure Nash equilibrium.}
\end{array} \]
Def. A game of perfect recall is an imperfect information game with the property that every play (= path from root to leaf) touches every information set at most once.

Here's a game of imperfect recall:

Games of imperfect recall are weird and badly behaved.

Perfect recall is often imposed as an extra assumption.